

Linear Program Application: *Aircraft Choices On Flight Routes Between JFK And SFO*

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1 Introduction

New York and San Francisco are the most important cities in east and west coasts, which makes the flight routes between these two cities are one the busiest routes above US continent. Due to the long distance and heavy volume of passenger, only large wide-body jets, such as Boeing 777, were capable for this routes before. However, with the development of technology, smaller narrow-body jets, such as A320, are also capable to take heavy volume of passenger on this kind of long haul routes. Therefore, we wonder what the choice for aircraft for a airlines on this route is that can optimize the profit from the round trip between JFK and SFO in a day. And the choices for aircraft are Boeing 787-8, 777-300ER, A330-300, and A320-200. The idea of this problem and data are from UBC Wiki.

2 Details of Linear Program

The objective for this Linear Program is to maximize the profit, so we can get $Z = Revenue - Cost$.

For the revenue, we only consider the income for passengers' tickets. Because the tickets depends on many variables such as the time, class, etc, we take the average price to calculate revenue. Meanwhile, the demand for flight from JFK to SFO is not identical with demand for flight for SFO to JFK, which results that the average price for flight from SFO to JFK is different with the price for flight from JFK to SFO. The details are in the Table 1.

Departure-Arrival	Average Price	Demand
JFK-SFO	500	850
SFO-JFK	400	1000

Table 1: Revenue Details

Therefore, from the graph, we can get the revenue from this round trip routes,

$$Reveune = 400 \times 1000 + 500 \times 850 = 825000$$

For cost, it also includes enormous amount of factors that influence it. To simply the computation, we summarize the cost per hour for each aircraft, and details are in the Table 2. Also to simply the computation, we take the average duration for the flight to 6 hours.

There are 8 variables in total, and details are in Table 3:

Aircraft	Seat Capacity	Total Variable Cost per hour(\$)
Boeing 787-8(788)	298	5303
Boeing 777-300ER(77W)	349	6751
Airbus A330-300(A333)	265	3716
Airbus A320-200(A322)	146	2845

Table 2: Aircraft Characteristics

a: No. of 788 SFO->JFK	b: No. of A322 SFO->JFK	c: No. of 77W SFO-> JFK	d: No. of A333 SFO-> JFK
e: No. of 788 JFK->SFO	f: No. of A322 JFK->SFO	g: No. of 77W JFK-> SFO	h: No. of A333 JFK-> SFO

Table 3: Variables Details

Notice that there are some constrains on these variables. It has to be non-negative, and if we have fractions in our final result, they need to be rounded up to the next integer.

Therefore, we have the objective Z such that:

$$Z = 825000 - 5303 * 6a - 2845 * 6b - 6751 * 6c - 3716 * 6d - 5303 * 6e - 2845 * 6f - 6751 * 6g - 3716 * 6h$$

With the objective clear, we get the constrains next. There are three kinds of constrains:

Flight Frequency

Because the demand for flight from passengers are distributed among the day, we need sufficient amount of flight that can distributed among different time among the day. Based on the demand, we need no less than 8 flights from SFO to JFK, and no less than 7 flights from JFK to SFO. Then we have the constrains:

$$\begin{aligned} a + b + c + d &\geq 8 \\ e + f + g + h &\geq 7 \end{aligned}$$

Supply of Seats:

According to the Table 1, we need to have aircraft that can take passengers no less than the demand for each route. Therefore, from the Table 2, we can get the constrains:

$$\begin{aligned} 298a + 146b + 349c + 265d &\geq 1000 \\ 298e + 146f + 349g + 265h &\geq 850 \end{aligned}$$

Aircraft Availability

Because we also need time for refueling, regular maintenance, loading and unloading passengers, etc., we assume that we can run our aircraft for up to 20 hours in a day. This constraint insures that there is a sufficient number of aircraft to cover the amount of time for all flights needed to cover the route with the frequency each route is flown.

$$\begin{aligned} 20(a + b + c + d) &\geq 8 * 6 \\ 20(e + f + g + h) &\geq 7 * 6 \end{aligned}$$

Notice that aircraft availability constrains can be rewrite to:

$$a + b + c + d \geq 2.4$$

$$e + f + g + h \geq 2.1$$

These constraints are included by the flight frequency constraints.

Notice that this problem is obviously feasible, because the airlines has to make a choice about the aircraft to run these routes. Otherwise, it won't have any customers, and the profit will be zero.

3 Simplex Algorithm

To solve this Linear Program, we try the Simplex Algorithm. Because this is a maximize problem, to put it in a equation form, we first convert the objective to minimize problem:

$$Z' = -Z = Cost - Revenue$$

$$Z' - 5303*6a - 2845*6b - 6751*6c - 3716*6d - 5303*6e - 2845*6f - 6751*6g - 3716*6h = -825000 = z_0$$

Next, we need to change inequalities to equality, so we add four slack variables $s, t, u, v \geq 0$, and we get:

$$298a + 146b + 349c + 265d - s = 1000$$

$$298e + 146f + 349g + 265h - t = 850$$

$$a + b + c + d - u = 8$$

$$e + f + g + h - v = 7$$

Therefore, we can get the canonical form:

$$\left[\begin{array}{cccccccccccccc} a & b & c & d & e & f & g & h & s & t & u & v & Z' & = \\ 298 & 146 & 349 & 265 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1000 \\ 0 & 0 & 0 & 0 & 298 & 146 & 349 & 265 & 0 & -1 & 0 & 0 & 0 & 850 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ -31818 & -17070 & -40506 & -22296 & -31818 & -17070 & -405067 & -22296 & 0 & 0 & 0 & 0 & 1 & -825000 \end{array} \right]$$

According to the canonical form, s, t, u, v have negative coefficients, so we need to make a feasible row, and we get:

$$\left[\begin{array}{cccccccccccccc} a & b & c & d & e & f & g & h & s & t & u & v & Z' & = \\ 298 & 146 & 349 & 265 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1000 \\ 0 & 0 & 0 & 0 & 298 & 146 & 349 & 265 & 0 & -1 & 0 & 0 & 0 & 850 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ -31818 & -17070 & -40506 & -22296 & -31818 & -17070 & -405067 & -22296 & 0 & 0 & 0 & 0 & 1 & -825000 \\ 299 & 147 & 350 & 266 & 299 & 147 & 350 & 266 & -1 & -1 & -1 & -1 & 0 & 1865 \end{array} \right]$$

Then we go to Phase 1. Because the computation is so complicated, we used Mathematica to help us with calculation for replacement. Then details of all computation will be in the appendix. With the replacements, we reach the end of Phase 1 and we get

$$\left[\begin{array}{cccccccccccccc} a & b & c & d & e & f & g & h & s & t & u & v & Z' & = \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ -152 & 0 & -203 & -119 & 0 & 0 & 0 & 0 & 1 & 0 & -146 & 0 & 0 & 168 \\ 0 & 0 & 0 & 0 & -152 & 0 & -203 & -119 & 0 & 1 & 0 & -146 & 0 & 172 \\ -14748 & 0 & -23436 & -5226 & -14748 & 0 & -23436 & -5226 & 0 & 0 & -17070 & -17070 & 1 & -568950 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Because the feasible row is all 0, this Linear Program is feasible. The nice thing is when we reduce feasible row to non-positive, we also make the objective row non-positive as well, which implies that we are in the solution form that satisfy Proposition 2.2a.

$$\left[\begin{array}{cccccccccccccc} a & b & c & d & e & f & g & h & s & t & u & v & Z' & = \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ -152 & 0 & -203 & -119 & 0 & 0 & 0 & 0 & 1 & 0 & -146 & 0 & 0 & 168 \\ 0 & 0 & 0 & 0 & -152 & 0 & -203 & -119 & 0 & 1 & 0 & -146 & 0 & 172 \\ -14748 & 0 & -23436 & -5226 & -14748 & 0 & -23436 & -5226 & 0 & 0 & -17070 & -17070 & 1 & -568950 \end{array} \right]$$

The basic vector is $(0, 8, 0, 0, 0, 7, 0, 0, 168, 172, 0, 0)$. The minimum Z' is -568950, which implies that the maximum Z is 568950 with $a = 0, b = 8, c = 0, d = 0, e = 0, f = 7, g = 0, h = 0$

4 Conclusion

According to the result from simplex algorithm, it is surprising that the optimal choice is to only use A322. This result is totally unbelievable in 30 years ago when these routes were dominated by large wide-body jets. The reason caused this result is that even though the wide-body jets can deliver more people with less cost in one time, which will lower that cost per person, the constrains from flight frequency cause that the wide-body jets may not be able to fully loaded, which leads to higher cost.

What also surprises me is that this is different with the real situation. Currently, even though the narrow-body jets, such as A320 and 737, dominate these trans-continent routes, there are still fairly amount flights that use large wide-body jets like 777. The reason is that the demand for flight varies from different time in a day. For example, the demand at early morning will be higher than the demand in early afternoon. With more passengers, it is easier to fullfil a large aircraft, so it will have lower cost than a smaller aircraft. However, our constrains does not consider this situation. Therefore, our linear program has some derivations from the real situation. However, if we can have more data that can espouse more details about the real situation, we can add more constrains to make our result closer to the reality.

References

- [1] Flight routes and linear programming
https://wiki.ubc.ca/Flight_routes_and_linear_programming_1

A Details of Replacement

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A = {{298, 146, 349, 265, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1000},
{0, 0, 0, 0, 298, 146, 349, 265, 0, -1, 0, 0, 0, 850},
{1, 1, 1, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 8},
{0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, -1, 0, 7},
{-5303, -2845, -6751, -3716, -5303, -2845, -6751, -3716, 0, 0, 0, 0, 1/6, -825000/6},
{299, 147, 350, 266, 299, 147, 350, 266, -1, -1, -1, -1, 0, 1865}
}
A[[5]] = A[[5]] * 6
```

$\{298, 146, 349, 265, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1000\}, \{0, 0, 0, 0, 298, 146, 349, 265, 0, -1, 0, 0, 0, 850\}, \{1, 1, 1, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0\}$
 $\{-31818, -17070, -40506, -22296, -31818, -17070, -40506, -22296, 0, 0, 0, 0, 1, -825000\}$

MatrixForm[A]

$$\begin{pmatrix} 298 & 146 & 349 & 265 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1000 \\ 0 & 0 & 0 & 0 & 298 & 146 & 349 & 265 & 0 & -1 & 0 & 0 & 0 & 850 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ -31818 & -17070 & -40506 & -22296 & -31818 & -17070 & -40506 & -22296 & 0 & 0 & 0 & 0 & 1 & -825000 \\ 299 & 147 & 350 & 266 & 299 & 147 & 350 & 266 & -1 & -1 & -1 & -1 & 0 & 1865 \end{pmatrix}$$

m1 = 1/349

$$\frac{1}{349}$$

R1 = A

$$\mathbf{R1}[[1]] = \mathbf{m1} * \mathbf{A}[[1]]$$

$$\mathbf{R1}[[3]] = \mathbf{R1}[[3]] - \mathbf{R1}[[1]]$$

$$\mathbf{R1}[[5]] = \mathbf{R1}[[5]] + 40506 * \mathbf{R1}[[1]]$$

$$\mathbf{R1}[[6]] = \mathbf{R1}[[6]] - 350 * \mathbf{R1}[[1]]$$

MatrixForm[R1]

$\{298, 146, 349, 265, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1000\}, \{0, 0, 0, 0, 298, 146, 349, 265, 0, -1, 0, 0, 0, 850\}, \{1, 1, 1, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0\}$

$$\left\{ \frac{298}{349}, \frac{146}{349}, 1, \frac{265}{349}, 0, 0, 0, 0, -\frac{1}{349}, 0, 0, 0, 0, \frac{1000}{349} \right\}$$

$$\left\{ \frac{51}{349}, \frac{203}{349}, 0, \frac{84}{349}, 0, 0, 0, 0, \frac{1}{349}, 0, -1, 0, 0, \frac{1792}{349} \right\}$$

$$\left\{ \frac{966306}{349}, -\frac{43554}{349}, 0, \frac{2952786}{349}, -31818, -17070, -40506, -22296, -\frac{40506}{349}, 0, 0, 0, 1, -\frac{247419000}{349} \right\}$$

$$\left\{ \frac{51}{349}, \frac{203}{349}, 0, \frac{84}{349}, 299, 147, 350, 266, \frac{1}{349}, -1, -1, -1, 0, \frac{300885}{349} \right\}$$

$$\begin{pmatrix} \frac{298}{349} & \frac{146}{349} & 1 & \frac{265}{349} & 0 & 0 & 0 & 0 & -\frac{1}{349} & 0 & 0 & 0 & 0 & \frac{1000}{349} \\ 0 & 0 & 0 & 0 & 298 & 146 & 349 & 265 & 0 & -1 & 0 & 0 & 0 & 850 \\ \frac{51}{349} & \frac{203}{349} & 0 & \frac{84}{349} & 0 & 0 & 0 & 0 & \frac{1}{349} & 0 & -1 & 0 & 0 & \frac{1792}{349} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ \frac{966306}{349} & -\frac{43554}{349} & 0 & \frac{2952786}{349} & -31818 & -17070 & -40506 & -22296 & -\frac{40506}{349} & 0 & 0 & 0 & 1 & -\frac{247419000}{349} \\ \frac{51}{349} & \frac{203}{349} & 0 & \frac{84}{349} & 299 & 147 & 350 & 266 & \frac{1}{349} & -1 & -1 & -1 & 0 & \frac{300885}{349} \end{pmatrix}$$

$$\mathbf{m2} = 1/298$$

$$\frac{1}{298}$$

$$\mathbf{R2} = \mathbf{R1}$$

$$\mathbf{R2}[[2]] = \mathbf{R1}[[2]] * \mathbf{m2}$$

$$\mathbf{R2}[[4]] = \mathbf{R2}[[4]] - \mathbf{R2}[[2]]$$

$$\mathbf{R2}[[5]] = \mathbf{R2}[[5]] + 31818 * \mathbf{R2}[[2]]$$

$$\mathbf{R2}[[6]] = \mathbf{R2}[[6]] - 299 * \mathbf{R2}[[2]]$$

$$\mathbf{MatrixForm}[\mathbf{R2}]$$

$$\left\{ \left\{ \frac{298}{349}, \frac{146}{349}, 1, \frac{265}{349}, 0, 0, 0, 0, -\frac{1}{349}, 0, 0, 0, 0, \frac{1000}{349} \right\}, \{0, 0, 0, 0, 298, 146, 349, 265, 0, -1, 0, 0, 0, 850\}, \left\{ \frac{51}{349}, \frac{203}{349}, 0, \frac{84}{349}, 0, 0, 0, 0, \frac{1}{349}, 0, 0, 0, 0, 0 \right\} \right\}$$

$$\left\{ 0, 0, 0, 0, 1, \frac{73}{149}, \frac{349}{298}, \frac{265}{298}, 0, -\frac{1}{298}, 0, 0, 0, \frac{425}{149} \right\}$$

$$\left\{ 0, 0, 0, 0, 0, \frac{76}{149}, -\frac{51}{298}, \frac{33}{298}, 0, \frac{1}{298}, 0, -1, 0, \frac{618}{149} \right\}$$

$$\left\{ \frac{966306}{349}, -\frac{43554}{349}, 0, \frac{2952786}{349}, 0, -\frac{220716}{149}, -\frac{483153}{149}, \frac{893781}{149}, -\frac{40506}{349}, -\frac{15909}{149}, 0, 0, 1, -\frac{32146026150}{52001} \right\}$$

$$\left\{ \frac{51}{349}, \frac{203}{349}, 0, \frac{84}{349}, 0, \frac{76}{149}, -\frac{51}{298}, \frac{33}{298}, \frac{1}{349}, \frac{1}{298}, -1, -1, 0, \frac{482690}{52001} \right\}$$

$$\begin{pmatrix} \frac{298}{349} & \frac{146}{349} & 1 & \frac{265}{349} & 0 & 0 & 0 & 0 & -\frac{1}{349} & 0 & 0 & 0 & 0 & \frac{1000}{349} \\ 0 & 0 & 0 & 0 & 1 & \frac{73}{149} & \frac{349}{298} & \frac{265}{298} & 0 & -\frac{1}{298} & 0 & 0 & 0 & \frac{425}{149} \\ \frac{51}{349} & \frac{203}{349} & 0 & \frac{84}{349} & 0 & 0 & 0 & 0 & \frac{1}{349} & 0 & -1 & 0 & 0 & \frac{1792}{349} \\ 0 & 0 & 0 & 0 & 0 & \frac{76}{149} & -\frac{51}{298} & \frac{33}{298} & 0 & \frac{1}{298} & 0 & -1 & 0 & \frac{618}{149} \\ \frac{966306}{349} & -\frac{43554}{349} & 0 & \frac{2952786}{349} & 0 & -\frac{220716}{149} & -\frac{483153}{149} & \frac{893781}{149} & -\frac{40506}{349} & -\frac{15909}{149} & 0 & 0 & 1 & -\frac{32146026150}{52001} \\ \frac{51}{349} & \frac{203}{349} & 0 & \frac{84}{349} & 0 & \frac{76}{149} & -\frac{51}{298} & \frac{33}{298} & \frac{1}{349} & \frac{1}{298} & -1 & -1 & 0 & \frac{482690}{52001} \end{pmatrix}$$

$$1000/349/146/349 < 1792/349/203/349$$

True

$$\mathbf{m3} = 349/146$$

$$\mathbf{R3} = \mathbf{R2}$$

$$\mathbf{R3}[[1]] = \mathbf{R2}[[1]] * \mathbf{m3}$$

$$\mathbf{R3}[[3]] = \mathbf{R3}[[3]] - \mathbf{R3}[[1]] * \frac{203}{349}$$

$$\mathbf{R3}[[5]] = \mathbf{R3}[[5]] + \mathbf{R3}[[1]] * \frac{43554}{349}$$

$$\mathbf{R3}[[6]] = \mathbf{R3}[[6]] - \mathbf{R3}[[1]] * \frac{203}{349}$$

MatrixForm[R3]

$$\frac{349}{146}$$

$$\left\{ \left\{ \frac{298}{349}, \frac{146}{349}, 1, \frac{265}{349}, 0, 0, 0, 0, -\frac{1}{349}, 0, 0, 0, 0, \frac{1000}{349} \right\}, \left\{ 0, 0, 0, 0, 1, \frac{73}{149}, \frac{349}{298}, \frac{265}{298}, 0, -\frac{1}{298}, 0, 0, 0, \frac{425}{149} \right\}, \left\{ \frac{51}{349}, \frac{203}{349}, 0, \frac{84}{349}, 0, 0, 0, 0, \frac{1}{349}, 0 \right\} \right.$$

$$\left. \left\{ \frac{149}{73}, 1, \frac{349}{146}, \frac{265}{146}, 0, 0, 0, 0, -\frac{1}{146}, 0, 0, 0, 0, \frac{500}{73} \right\} \right.$$

$$\left. \left\{ -\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, 0, 0, 0, 0, \frac{1}{146}, 0, -1, 0, 0, \frac{84}{73} \right\} \right.$$

$$\left. \left\{ \frac{220716}{73}, 0, \frac{21777}{73}, \frac{634167}{73}, 0, -\frac{220716}{149}, -\frac{483153}{149}, \frac{893781}{149}, -\frac{8535}{73}, -\frac{15909}{149}, 0, 0, 1, -\frac{6714656550}{10877} \right\} \right.$$

$$\left. \left\{ -\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, 0, \frac{76}{149}, -\frac{51}{298}, \frac{33}{298}, \frac{1}{146}, \frac{1}{298}, -1, -1, 0, \frac{57630}{10877} \right\} \right.$$

$$\left(\begin{array}{cccccccccccccc} \frac{149}{73} & 1 & \frac{349}{146} & \frac{265}{146} & 0 & 0 & 0 & 0 & -\frac{1}{146} & 0 & 0 & 0 & 0 & \frac{500}{73} \\ 0 & 0 & 0 & 0 & 1 & \frac{73}{149} & \frac{349}{298} & \frac{265}{298} & 0 & -\frac{1}{298} & 0 & 0 & 0 & \frac{425}{149} \\ -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & 0 & 0 & 0 & 0 & \frac{1}{146} & 0 & -1 & 0 & 0 & \frac{84}{73} \\ 0 & 0 & 0 & 0 & 0 & \frac{76}{149} & -\frac{51}{298} & \frac{33}{298} & 0 & \frac{1}{298} & 0 & -1 & 0 & \frac{618}{149} \\ \frac{220716}{73} & 0 & \frac{21777}{73} & \frac{634167}{73} & 0 & -\frac{220716}{149} & -\frac{483153}{149} & \frac{893781}{149} & -\frac{8535}{73} & -\frac{15909}{149} & 0 & 0 & 1 & -\frac{6714656550}{10877} \\ -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & 0 & \frac{76}{149} & -\frac{51}{298} & \frac{33}{298} & \frac{1}{146} & \frac{1}{298} & -1 & -1 & 0 & \frac{57630}{10877} \end{array} \right)$$

$$\mathbf{R3}[[2, 14]]/\mathbf{R3}[[2, 6]] < \mathbf{R3}[[4, 14]]/\mathbf{R3}[[4, 6]]$$

True

$$\mathbf{m4} = 1/\mathbf{R3}[[2, 6]]$$

$$\frac{149}{73}$$

$$\mathbf{R4} = \mathbf{R3}$$

$$\mathbf{R4}[[2]] = \mathbf{R3}[[2]] * \mathbf{m4}$$

$$\mathbf{R4}[[4]] = \mathbf{R4}[[4]] - 76/149 * \mathbf{R4}[[2]]$$

$$\mathbf{R4}[[5]] = \mathbf{R4}[[5]] + \frac{220716}{149} * \mathbf{R4}[[2]]$$

$$\mathbf{R4}[[6]] = \mathbf{R4}[[6]] - 76/149 * \mathbf{R4}[[2]]$$

MatrixForm[R4]

$$\left\{ \left\{ \frac{149}{73}, 1, \frac{349}{146}, \frac{265}{146}, 0, 0, 0, 0, -\frac{1}{146}, 0, 0, 0, 0, \frac{500}{73} \right\}, \left\{ 0, 0, 0, 0, 1, \frac{73}{149}, \frac{349}{298}, \frac{265}{298}, 0, -\frac{1}{298}, 0, 0, 0, \frac{425}{149} \right\}, \left\{ -\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, 0, 0, 0, 0, \frac{1}{146} \right\} \right.$$

$$\left. \left\{ 0, 0, 0, 0, \frac{149}{73}, 1, \frac{349}{146}, \frac{265}{146}, 0, -\frac{1}{146}, 0, 0, 0, \frac{425}{73} \right\} \right.$$

$$\left. \left\{ 0, 0, 0, 0, -\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, 0, \frac{1}{146}, 0, -1, 0, \frac{86}{73} \right\} \right.$$

$$\left. \left\{ \frac{220716}{73}, 0, \frac{21777}{73}, \frac{634167}{73}, \frac{220716}{73}, 0, \frac{21777}{73}, \frac{634167}{73}, -\frac{8535}{73}, -\frac{8535}{73}, 0, 0, 1, -\frac{44435250}{73} \right\} \right.$$

$$\left. \left\{ -\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, -\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, \frac{1}{146}, \frac{1}{146}, -1, -1, 0, \frac{170}{73} \right\} \right.$$

$$\left(\begin{array}{cccccccccccccc} \frac{149}{73} & 1 & \frac{349}{146} & \frac{265}{146} & 0 & 0 & 0 & 0 & -\frac{1}{146} & 0 & 0 & 0 & 0 & \frac{500}{73} \\ 0 & 0 & 0 & 0 & \frac{149}{73} & 1 & \frac{349}{146} & \frac{265}{146} & 0 & -\frac{1}{146} & 0 & 0 & 0 & \frac{425}{73} \\ -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & 0 & 0 & 0 & 0 & \frac{1}{146} & 0 & -1 & 0 & 0 & \frac{84}{73} \\ 0 & 0 & 0 & 0 & -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & 0 & \frac{1}{146} & 0 & -1 & 0 & \frac{86}{73} \\ \frac{220716}{73} & 0 & \frac{21777}{73} & \frac{634167}{73} & \frac{220716}{73} & 0 & \frac{21777}{73} & \frac{634167}{73} & -\frac{8535}{73} & -\frac{8535}{73} & 0 & 0 & 1 & -\frac{44435250}{73} \\ -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & \frac{1}{146} & \frac{1}{146} & -1 & -1 & 0 & \frac{170}{73} \end{array} \right)$$

$$\mathbf{m5} = 146$$

$$\mathbf{R5} = \mathbf{R4}$$

$$\mathbf{R5}[[4]] = \mathbf{R4}[[4]] * \mathbf{m5}$$

$$\mathbf{R5}[[2]] = \mathbf{R5}[[2]] + 1/146 * \mathbf{R5}[[4]]$$

$$\mathbf{R5}[[5]] = \mathbf{R5}[[5]] + \frac{8535}{73} * \mathbf{R5}[[4]]$$

$$\mathbf{R5}[[6]] = \mathbf{R5}[[6]] - 1/146 * \mathbf{R5}[[4]]$$

MatrixForm[R5]

146

$$\left\{ \left\{ \frac{149}{73}, 1, \frac{349}{146}, \frac{265}{146}, 0, 0, 0, 0, -\frac{1}{146}, 0, 0, 0, 0, \frac{500}{73} \right\}, \left\{ 0, 0, 0, 0, \frac{149}{73}, 1, \frac{349}{146}, \frac{265}{146}, 0, -\frac{1}{146}, 0, 0, 0, \frac{425}{73} \right\}, \left\{ -\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, 0, 0, 0, 0, \frac{1}{146} \right\} \right.$$

$$\left. \left\{ 0, 0, 0, 0, -152, 0, -203, -119, 0, 1, 0, -146, 0, 172 \right\} \right.$$

$$\left. \left\{ 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, -1, 0, 7 \right\} \right.$$

$$\left. \left\{ \frac{220716}{73}, 0, \frac{21777}{73}, \frac{634167}{73}, -14748, 0, -23436, -5226, -\frac{8535}{73}, 0, 0, -17070, 1, -\frac{42967230}{73} \right\} \right.$$

$$\left\{-\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, 0, 0, 0, 0, \frac{1}{146}, 0, -1, 0, 0, \frac{84}{73}\right\}$$

$$\begin{pmatrix} \frac{149}{73} & 1 & \frac{349}{146} & \frac{265}{146} & 0 & 0 & 0 & 0 & -\frac{1}{146} & 0 & 0 & 0 & 0 & \frac{500}{73} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & 0 & 0 & 0 & 0 & \frac{1}{146} & 0 & -1 & 0 & 0 & \frac{84}{73} \\ 0 & 0 & 0 & 0 & -152 & 0 & -203 & -119 & 0 & 1 & 0 & -146 & 0 & 172 \\ \frac{220716}{73} & 0 & \frac{21777}{73} & \frac{634167}{73} & -14748 & 0 & -23436 & -5226 & -\frac{8535}{73} & 0 & 0 & -17070 & 1 & -\frac{42967230}{73} \\ -\frac{76}{73} & 0 & -\frac{203}{146} & -\frac{119}{146} & 0 & 0 & 0 & 0 & \frac{1}{146} & 0 & -1 & 0 & 0 & \frac{84}{73} \end{pmatrix}$$

$$\mathbf{m6} = 146$$

$$\mathbf{R6} = \mathbf{R5}$$

$$\mathbf{R6}[[3]] = \mathbf{R5}[[3]] * \mathbf{m6}$$

$$\mathbf{R6}[[1]] = \mathbf{R6}[[1]] + 1/146 * \mathbf{R6}[[3]]$$

$$\mathbf{R6}[[5]] = \mathbf{R6}[[5]] + \frac{8535}{73} * \mathbf{R6}[[3]]$$

$$\mathbf{R6}[[6]] = \mathbf{R6}[[6]] - 1/146 * \mathbf{R6}[[3]]$$

$$\mathbf{MatrixForm}[\mathbf{R6}]$$

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$$\left\{\left\{\frac{149}{73}, 1, \frac{349}{146}, \frac{265}{146}, 0, 0, 0, 0, -\frac{1}{146}, 0, 0, 0, 0, \frac{500}{73}\right\}, \{0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, -1, 0, 7\}, \left\{-\frac{76}{73}, 0, -\frac{203}{146}, -\frac{119}{146}, 0, 0, 0, 0, \frac{1}{146}, 0, -1, 0, 0, \frac{84}{73}\right\}\right\}$$

$$\{-152, 0, -203, -119, 0, 0, 0, 0, 1, 0, -146, 0, 0, 168\}$$

$$\{1, 1, 1, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 8\}$$

$$\{-14748, 0, -23436, -5226, -14748, 0, -23436, -5226, 0, 0, -17070, -17070, 1, -568950\}$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ -152 & 0 & -203 & -119 & 0 & 0 & 0 & 0 & 1 & 0 & -146 & 0 & 0 & 168 \\ 0 & 0 & 0 & 0 & -152 & 0 & -203 & -119 & 0 & 1 & 0 & -146 & 0 & 172 \\ -14748 & 0 & -23436 & -5226 & -14748 & 0 & -23436 & -5226 & 0 & 0 & -17070 & -17070 & 1 & -568950 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$